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Superconducting billiard cavities with chaotic dynamics: An experimental test of statistical measures

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The eigenmodes of a superconducting, two-dimensional hyperbolically shaped cavity have been measured for frequencies below 20 GHz. These data of high accuracy, along with data from our previous measurement of a quarter stadium billiard [Gräf *et al.*, Phys. Rev. Lett. **64**, 1296 (1992)], are used as a test of three models (Brody, Berry and Robnik, Lenz and Haake) of distributions of eigenmodes. It becomes clear that data spanning universal and nonuniversal regimes and analyses extending at least up to two-level measures are essential to draw firm conclusions about the model descriptions.

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In recent years, signatures of quantum chaos have been the subject of intense research by several groups [1–13]. It is widely recognized that chaotic behavior of a system reveals itself in the distribution of its eigenmodes. For pure systems, in general, the description of regular and chaotic behavior is given as Poissonian and Gaussian ensembles, respectively, though non-Poissonian type regular systems also occur [14]. The situation is much less clear for mixed systems, i.e., systems which are neither purely regular nor completely chaotic; here a question of principal interest is their degree of chaoticity.

For mixed systems the following descriptions for the nearest-neighbor spacing distributions (NND's) are available in the literature. Brody *et al.* [15] provided an empirical formula

$$P(s)=c_1 s^{\omega}\exp(-c_2 s^{(\omega+1)}), \tag{1}$$

where s is the normalized spacing, c_1 and c_2 are normalization constants, and ω is the so called Brody parameter. No clear-cut physical meaning can be attributed to ω. The Berry-Robnik distribution [16] results from the assumption that the system is a composite of two distinct, noninteracting phase spaces. It is given by

$$P(s)=(1-q)^2 e^{(q-1)s}\{\mathrm{erfc}(\tfrac{1}{2}\sqrt{\pi}qs)+[2q(1-q)+\tfrac{1}{2}\pi q^3 s]e^{(1/4)\pi q^2 s^2}\}, \tag{2}$$

where q represents the chaotic fraction of the total phase space. Finally, Lenz and Haake [17] consider the Hamiltonian of the system as made up of a regular and a chaotic component which renders the entire phase space chaotic to some extent. Their distribution can be written in terms of the modified Bessel function $I_0(x)$ and the Tricomi function $U(a,c,x)$ as

$$P(s)=\frac{su(\lambda)^2}{\lambda}\exp\left(\frac{-u(\lambda)^2 s^2}{4\lambda^2}\right)\times\int_0^{\infty}e^{-(\xi^2+2\xi\lambda)}I_0\left(\frac{s\xi u(\lambda)}{\lambda}\right)d\xi, \tag{3}$$

where $u(\lambda)=\sqrt{\pi}U(-\frac{1}{2},0,\lambda^2)$, and the parameter λ is a measure for the relative influence of the chaotic contribution to the system.

Thus each distribution is characterized by a single mixing parameter: ω for Brody, q for Berry-Robnik, and λ for Lenz-Haake. We renormalize the Lenz-Haake parameter by $\Lambda=\lambda/\sqrt{\lambda^2+1}$. When the parameters ω, q, Λ are zero, one has pure Poissonian distributions whereas for parameters of unit magnitude Wigner distributions are obtained.

It is of considerable interest to verify if one is able to distinguish among these distributions and to examine the sensitivities of different statistical measures. To this end, we exploit the precision of high resolution measurements attainable in superconducting microwave resonators. We should stress that in this work simultaneous experimental analyses of mixed and nearly pure systems are carried out. Presently, however, only the Berry-Robnik distribution is amenable to calculations of second order measures [18] such as the number variance Σ^2 and the Dyson-Mehta statistics Δ_3, while the other two distributions are limited to NND $[P(s)]$ formulas.

Previously, we reported the measurements on a two-dimensional quarter of a stadium shaped superconducting rf cavity [19], a mixed system, with a Q value corresponding to a frequency resolution of $\Delta f/f \approx 10^{-5}-10^{-7}$, installed in the cryostat of the Superconducting Darmstadt Linear Electron Accelerator (S-DALINAC) [20] operating at a temperature of 2 K. In the meantime the results of these measurements have been the subject of several studies [14,21], including a very extensive semiclassical analysis confirming that the spectrum of the quarter stadium is influenced by the existence of a nonisolated periodic orbit, the "bouncing ball" orbit. It is therefore used as a test model for the present analysis within the above mentioned simple models. Since then, we measured for comparison the eigenmodes of an almost purely chaotic system, a hyperbola billiard, under the same experimental conditions. The geometries of both billiards are sketched in Fig. 1. The hyperbola billiard has a straight edge along the x axis and is of length 1350 mm. The width y of the cavity is shaped as $y=x$ for $x \leq 186$ mm and $y=(34\,596/x)$ mm for $186 \leq x \leq 1321$ mm and capped with a quarter circle of 23 mm radius. This geometry assures that only isolated periodic orbits exist in the x-y plane. The third dimension (cavity height) is 7 mm to render the cavity a two-dimensional resonator for frequencies below 20 GHz

Details of the measurements and identification of eigenmodes are to be found in Ref. [19]. The system resolution was more than adequate to locate the 1051 eigenmodes of the hyperbola billiard and ensured that there were no missing modes. Clearly, this became possible only due to the availability of superconducting cavities. Since the spectrum of the stadium contains 1060 modes, both statistical samples are of the same size.

Unlike in Ref. [19] where we treated the bouncing ball orbits and chaotic orbits as separate entities, here we consider the system as a whole. In the following, we discuss the results of the measurements in the framework of the models mentioned above. The first step in the analyses is a fit of the NND for the two billiards. As shown in the upper part of Fig. 1, for the three distributions the parameters ω, q, $\Lambda \approx 1.0$, i.e., they are consistent with a purely chaotic system for the

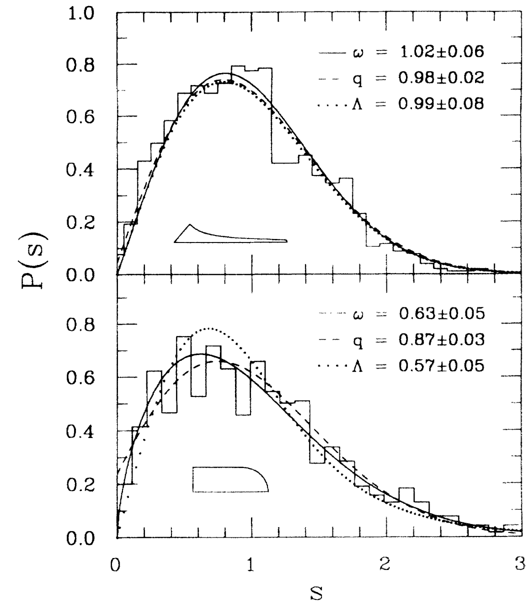

FIG. 1. Nearest-neighbor spacing distributions. The histograms correspond to the measured data for the hyperbola (upper part) and the stadium billiard (lower part). The full, dashed, and dotted lines show the best fit of the Brody, Berry-Robnik, and Lenz-Haake distributions, respectively, with the mixing parameters given in the figures. The billiards are sketched to illustrate their geometry. Note that the histogram for the stadium billiard is not the same as in the upper part of Fig. 4 of Ref. [19] because here contributions from the bouncing ball orbits are still included.

hyperbola billiard. In the case of the stadium billiard, however, the fits yield very different parameter values, as indicated in the lower part of Fig. 1, viz., $\omega=0.63\pm0.05$, $q=0.87\pm0.03$, and $\Lambda=0.57\pm0.05$ for the Brody, Berry-Robnik, and Lenz-Haake distributions, respectively. The Berry-Robnik fit needs a rather large chaotic component ($q=0.87$), whereas the Lenz-Haake parameter implies that the chaotic and regular components are of amplitudes $\Lambda=0.57$ and $\sqrt{1-\lambda^2}=0.82$, respectively. This feature can, perhaps, be intuitively understood if a small admixture of a chaotic component influences a coherent superposition of regular and chaotic components much more strongly than in the case of incoherent superposition. All three distributions describe the experimental data equally well (as seen from the χ^2 values), therefore no distinction between the three models is possible from the NND.

Further analyses are required to verify which, if any, of the three models are correct descriptions of the systems under study. Unfortunately, at least at this time, only the Berry-Robnik model can be tested for two-level measures, such as spectral rigidity $\Delta_3(L)$ and number variance $\Sigma^2(L)$ statistics. Figure 2 shows the experimental Δ_3 statistics for the hyperbola and stadium billiards plotted against L/L_{max}, where L_{max} is the saturation length [22] for easy comparison. For our cases the saturation lengths are $L_{max}=6$ and 15 for the hyperbola and stadium billiards, respectively. In case of a mixed system, the Δ_3 statistics follows the GOE prediction for small L values. It then steadily deviates from GOE saturation $\Delta_{3,GOE}^{sat}$ for $L>L_{max}$ and reaches a higher saturation

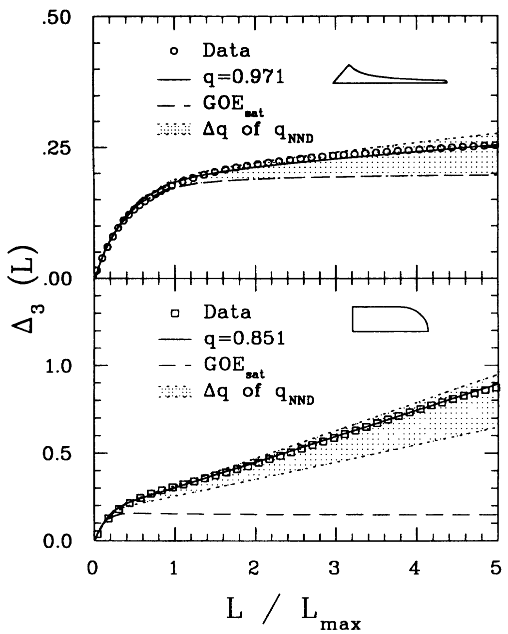

FIG. 2. The Δ_3 statistics as a function of L/L_{max}. The upper part shows the data of the hyperbola billiard. The fit of the Berry-Robnik model (full line) describes the data well, the dashed line gives the saturated GOE part contained in the fit. For demonstration of the sensitivity, the limits of the mixing parameter q extracted from the NND (see Fig. 1) are given by the shaded area. The lower part illustrates the situation for the stadium billiard. Note that the experimental Δ_3 values have been derived according to Eq. (3.30) of Ref. [6].

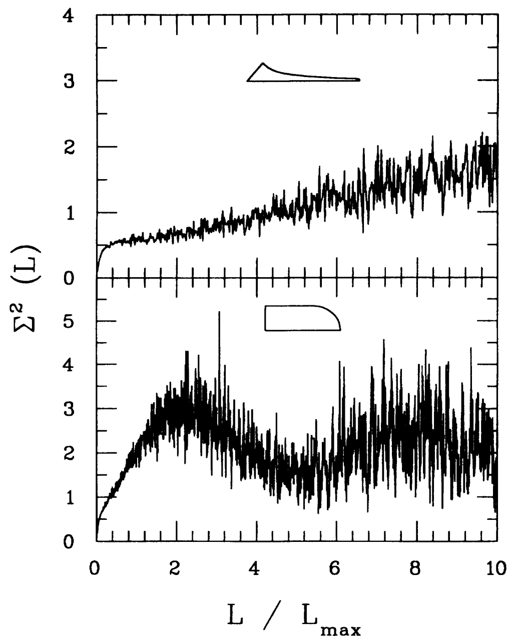

FIG. 3. The Σ^2 statistics as a function of L/L_{max}. The linear increase for $L/L_{max}>2$ is the sensitive signature of the small regular part (3%) in case of the hyperbola billiard. For the stadium billiard large oscillations appear as a signature for nonuniversal behavior in the region $L/L_{max}>2$.

value. In the Berry-Robnik description of mixed systems, we have $\Delta_3(L)=\Delta_{3,P}((1-q)L)+\Delta_{3,GOE}(qL)$ where q is the mixing parameter. For the hyperbola billiard, Δ_3 follows the trend for a GOE system, though a small ($\approx 3\%$) regular component is visible and a mixing parameter of $q=0.971$ has been determined from the linear increase in Δ_3 in the range $2\leq L/L_{max}\leq 5$. The resulting Δ_3 then describes the experimental data rather well, as can be seen in Fig. 2. The saturation value for the GOE part turns out to be $\Delta_{3,GOE}^{sat}=0.198$, significantly smaller than the theoretical prediction [22] of $\Delta_{3,GOE}^{sat}=0.344$ for the hyperbola billiard.

The shaded area represents the spread in Δ_3 due to the q parameter fit of the NND. The upper limit of $q=1$ coincides with $\Delta_{3,GOE}^{sat}$. The analysis for the stadium billiard was performed exactly as in the case of the hyperbola, yielding a mixing parameter of $q=0.851$ in agreement with the value deduced from the NND. It is noteworthy that the Δ_3 statistics enables us to determine the mixing parameter much more accurately than it is possible by the NND. Again, the experimental data are well reproduced, but only if an unreasonably small value of $\Delta_{3,GOE}^{sat}=0.148$ is used, compared to the predicted value of 0.437. On the other hand, our previous analysis, subtracting out the bouncing ball contribution, showed that the remainder is a pure GOE, saturating at ~ 0.33. Clearly there is a contradiction here. This is not too surprising since the semiclassical analysis of the stadium billiard [19] shows that the regular contribution to the level statistics (mainly resulting from the existence of the bouncing ball orbit) does not show a pure Poissonian behavior (see Fig. 7 of Ref. [14]), a property not incorporated in the simple model of Berry and Robnik.

We tested the data against another two-level measure, viz., the number variance $\Sigma^2(L)$. The experimental results are plotted in Fig. 3. The regular component in the hyperbola billiard is much more apparent in this plot since Σ^2 shows a linear rise above the GOE saturation for $L/L_{max}\geq 2$. The sensitivity of this measure to small admixtures ($\approx 3\%$) is quite impressive. For the stadium billiard, Σ^2 shows a strong oscillatory pattern for $L/L_{max}\geq 2$. This has appeared in numerical studies [23] but here we have observed it experimentally. It is a clear indication that the data comprise both universal and nonuniversal regimes. Thus they cannot be described by a simple superposition of Poissonian and Gaussian components as in the case of Δ_3. In passing we note that, to our knowledge, there are at present no theoretical prescriptions to evaluate the higher order moments for mixed systems. But it is unlikely that three- and four-point measures [6] like skewness $\gamma_1(L)$ and excess $\gamma_2(L)$ provide extra sensitivity to the level statistics of the mixed system. The GOE and Poisson estimates for these measures differ significantly from each other only for $L\approx 1$. For both large and small L, the two-level distributions give nearly equal values for $\gamma_1(L)$ and $\gamma_2(L)$. For the present mixed system, significant deviations from both GOE and Poisson predictions occur at $L/L_{max}\geq 2$, for which $\gamma_1,\gamma_2\approx 0$ for both cases.

From the present investigation of hyperbola and stadium billiards, which constitute an almost pure and a strongly mixed system, respectively, the following conclusions may be drawn: Nearest-neighbor spacing distributions are not adequate measures in order to verify a model. We conclude this from the fact that three models differing in the details of mixing reproduce the experimental data equally well. Only two-level measures, viz., Δ_3 and Σ^2 statistics, reveal the deficiency of a model. Though the Δ_3 statistics is capable of revealing small admixtures, it fails to identify the character of the mixing. It became possible to examine these details as the present experimental data sets extend over the universal and nonuniversal regimes of the system dynamics. As a result we remark that it is highly desirable to parametrize the two-level measures for models describing mixed systems such as the model by Lenz and Haake and later ones [12] based on random matrices.

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